NECESSARY AND SUFFICIENT CONDITIONS FOR LTI REPRESENTATIONS OF ADAPTIVE SYSTEMS WITH SINUSOIDAL REGRESSORS

David S. Bayard

Jet Propulsion Laboratory
California Institute of Technology
4800 Oak Grove Drive
Pasadena, CA 91109

Abstract

This paper establishes necessary and sufficient conditions for an adaptive system with a sinusoidal regressor (i.e., a regressor comprised exclusively of sinusoidal signals) to admit an exact linear time-invariant (LTI) representation. These conditions are important because a large number of adaptive systems used in practice have sinusoidal regressors, and the stability, convergence and robustness properties of systems having LTI representations can be completely analyzed by well-known methods.

1 INTRODUCTION

A large number of adaptive systems used in practice (e.g., for adaptive signal processing, noise cancelling, acoustics, vibration suppression, etc.), have regressors which contain sinusoidal excitations. In certain interesting cases, such systems have been found to admit exact finite-dimensional linear time-invariant (LTI) representations. Such cases are important because in contrast to nonlinear and/or time-varying representations, the stability, convergence and robustness properties of LTI systems can be completely characterized using standard methods.

The use of LTI representations for analysis of adaptive feedforward systems can be found predominantly in papers from the acoustics and signal processing community. This includes the pioneering work of Glover [12] and later extensions found in Morgan and Sanford [20], Morgan [21], Elliott *et. al.* [10], and Widrow and Stearns [34]. In the control community, these LTI representations have found their way into certain specialized applications such as adaptive helicopter rotor control (cf., Shaw and Albion [27], Hall and Wereley [13]), adaptive vibration control (cf., Bodson *et. al.* [4], Messner and Bodson [17]), and adaptive structural control and active isolation (cf., Spanos and Rahman [30], Collins [7], Sievers and von Flotow [28]). In each of these papers, the authors were able to analyze their application more thoroughly within an LTI framework than would have been possible if they had used

Lyapunov analysis, Hyperstability, or other nonlinear/time-varying tools typically associated with adaptive control analysis.

Interestingly, despite various successes in specific application areas, no general unified theory of LTI adaptive feedforward systems has emerged. In particular, no definitive conditions for the LTI phenomena have been previously established.

In this paper, a general unified theory of LTI representations is developed for adaptive systems having harmonic regressors. The main result is a precise condition (i.e., both necessary and sufficient), for such harmonic adaptive systems to have an exact LTI representation, and a clod-form analytic expression for this LTI representation when the condition is satisfied. The theory completely unifies existing results by reproducing as special cases all known instances of LTI adaptive systems found in the literature. More importantly, the theory generalizes existing results by indicating a much larger class of LTI adaptive systems than previously known. Examples are given to demonstrate the usefulness and implications of the result. All results in this paper are based on the analysis in a recent report [2].

2 BACKGROUND

2.1 Adaptive Systems with Harmonic Regressors

The configuration to be studied is shown in Figure 2.1. An estimate $\hat{y}$ of some signal y is to be constructed as a linear combination of the elements of a regressor vector $x(t) \in R^N$, i.e.,

Estimated Signal

$$\hat{y} = w(t)^T x(t) \tag{2.1}$$

where $w(t) \in R^N$ is a parameter vector which is tuned in real-time using the adaptation algorithm,

Adaptation Algorithm

$$w = \mu \Gamma(p) [\hat{x}(t) e(t)] \tag{2.2}$$

Here, the notation $\Gamma(p)[\cdot]$ is used to denote the multivariable LTI transfer function $\Gamma(s) \cdot I$ where $\Gamma(s)$ is any LTI transfer function in the Laplace s operator (the differential operator p will replace the Laplace operator s in all time-domain filtering expressions); the term $e(t) \in R^1$ is an error signal; $\mu > 0$ is an adaptation gain; and the signal $\hat{x}$ is obtained by filtering the regressor x through any stable filter $F(p)$, i.e.,

Regressor Filtering

$$\hat{x} = F(p)[x] \tag{2.3}$$

The notation $F(p)[\cdot]$ denotes the multivariable LTI transfer function $F(s) \cdot I$ with SISO filter $F(s)$, acting on the indicated vector time domain signal.

For the purposes of this paper, it will be assumed that the regressor x can be written as a linear combination of m distinct sinusoidal components $\{\omega_i\}_{i=1}^{m}$, $0 < \omega_1 < \omega_2 < \ldots < \omega_m$, where the frequencies have been ordered by size from smallest to largest. Equivalently, it is assumed that there exists a matrix $\mathcal{X} \in R^{N\times 2m}$ such that,

HARMONIC *Regressor*

$$x = \mathcal{X}c(t) \tag{2.4}$$

$$c(t) = [\sin(\omega_1 t), \cos(\omega_1 t), ..., \sin(\omega_m t), \cos(\omega_m t)]^T \in R^{2m} \tag{2.5}$$

Equations (2.1)-(2.5) taken together will be referred to as a *harmonic adaptive system*. Collectively, these equations define an important open-loop mapping from the error signal e to the estimated output $\hat{y}$. Because of its importance, this mapping will be denoted by the special character $\mathcal{H}$, i.e.,

$$\hat{y} = \mathcal{H}[e] \tag{2.6}$$

The special structure of $\mathcal{H}$ is depicted in Figure 2.1.

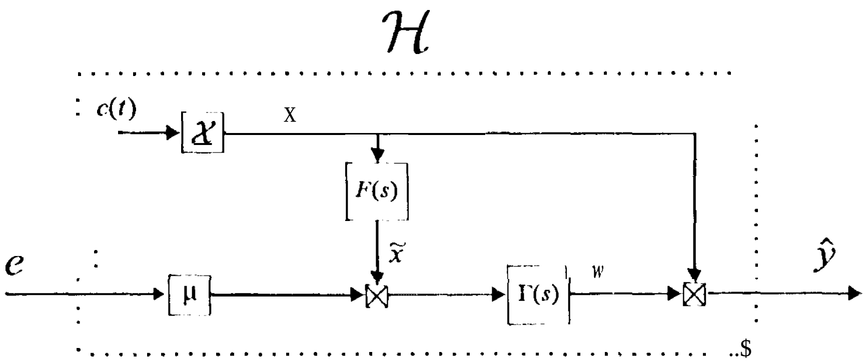


Figure 2.1: LTV operator $\hat{y} = \mathcal{H}[e]$ for adaptive system with harmonic regressor x, adaptation law $\Gamma(s)$, and regressor filter $F(s)$

Most generally $\mathcal{H}$ is a linear time-varying (LTV) operator. However, the main results of this paper show that under certain simple conditions on the matrix $\mathcal{X}$, the mapping $\mathcal{H}$ is actually linear time-invariant (LTI). This result has profound implications for many classes of adaptive systems, since they can be designed and analyzed completely using LTI theory.

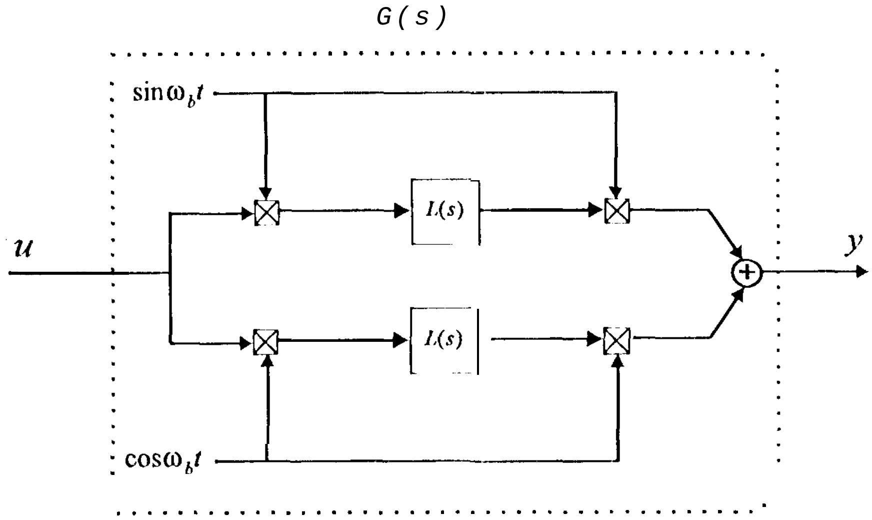
REMARK 2.1 The definition of $\Gamma(s)$ is left intentionally general to include analysis of the gradient algorithm (i.e., with the choice $\Gamma(s) = 1/s$), the gradient algorithm with leakage (i.e., $\Gamma(s) = 1/(s+\sigma)$; $\sigma \geq 0$), proportional-plus-integral adaptation (i.e., $\Gamma(s) = k_p + k_i/s$), or arbitrary linear adaptation algorithms of the designer's choosing. Adaptation laws which are nonlinear or normalized (e.g., divided by the norm of the regressor), are not considered here since they do not have an equivalent LTI representation $\Gamma(s)$. ■

REMARK 2.2 The use of the regressor filter $F(s)$ is (2.3) allows the unified treatment of many important adaptation algorithms including the well-known Filtered-X algorithm from the signal processing literature [32][21][5][33][16], and the Augmented Error algorithm of Monopoli [19]. Since x is comprised purely of sinusoidal components and F in (2.3) is stable, all subsequent analysis will assume that the filter output $\hat{x}$ has reached a steady-state condition. ■

2.2 Discussion

Most generally $\mathcal{H}$ in (2.6) is a *linear time-varying* (LTV) operator. However, under certain conditions on the matrix $\mathcal{X}$, the mapping $\mathcal{H}$ is actually *linear time-invariant.*

The intuition behind this seemingly strange phenomena is explained by the modulation/demodulation properties of multiplicative sinusoidal terms. As a simple example, consider the LTI bandpass filter (BPF) implementation shown in Figure 2.2.



$$G(s) = \frac{1}{2}\left(L(s - j\omega_b) + L(s + j\omega_b)\right)$$

Figure 2.2: Exact LTI Bandpass filter $y = G(p)u$ using Lowpass filter $L(s)$ and modulation properties of sandwiched sinusoidal multiplications

Here, a lowpass filter $L(s)$ is sandwiched between matched sine/cosine multiplications. By inspection, the output can be written in terms of convolution integrals as,

$$y = \sin\omega_b t \int_0^t \ell(\tau)\sin\omega_b(t-\tau)u(t-\tau)d\tau + \cos\omega_b t \int_0^t \ell(\tau)\cos\omega_b(t-\tau)u(t-\tau)d\tau \qquad (2.7)$$

where $\ell(t)$ is the impulse response of the low-pass filter $L(s)$. At first glance this looks like an LTV system. However, substituting the trigonometric identity,

$$\sin \omega_b t \sin \omega_b(t - \tau) + \cos \omega_b t \cos \omega_b(t - \tau) = \cos \omega_b \tau \tag{2.8}$$

into (2.7) and rearranging gives,

$$y = \int_0^t \ell(\tau) \cos \omega_b(\tau) u(t - \tau) d\tau \tag{2.9}$$

This integral can be recognized as a convolution of the input u with the *time-invariant* impulse response $\ell(t) \cos \omega_b(t)$. Hence, the overall filter is LTI even though it has time-varying elements. The essential relation is identity (2.8) which indicates that the function of both t and τ on the left hand side, can be written purely as the function of τ seen on the right hand side.

It is also worth noting that the impulse response of the convolution integral (2.9) is formed by modulating the lowpass filter response $\ell(t)$ by $\cos(\omega_b t)$, so that the resulting LTI filter has the bandpass characteristic,

$$L'(s) = \mathcal{L}\{\ell(t) \cos \omega_b t\} = \frac{1}{2}(L(s - j\omega_b) + L(s + j\omega_b)) \tag{2.10}$$

Here we have used the well-known modulation property $\mathcal{L}\{\ell(t) e^{j\omega_b t}\} = L(s - j\omega_b)$ of the Laplace transform [3].

As a specific example, let $L(s) = 1/(s + a)$ in Figure 2.2. Then the operator from u to y shown in Figure 2.2 is exactly representable as an LTI filter, and has a (bandpass) transfer function which can be computed from (2.10) as,

$$G(s) = \frac{s + a}{(s + a)^2 + \omega_b^2} \tag{2.11}$$

3 LTI REPRESENTATIONS

3.1 Single-Tone Regressor Case

Lemma 3.1 first characterizes the LTI operator for the case of a single tone regressor. The arrangement is depicted in Figure 3.1 and corresponds to the special Case of $\mathcal{X} = d_i \cdot I_{2\times 2}$ in (2.4).

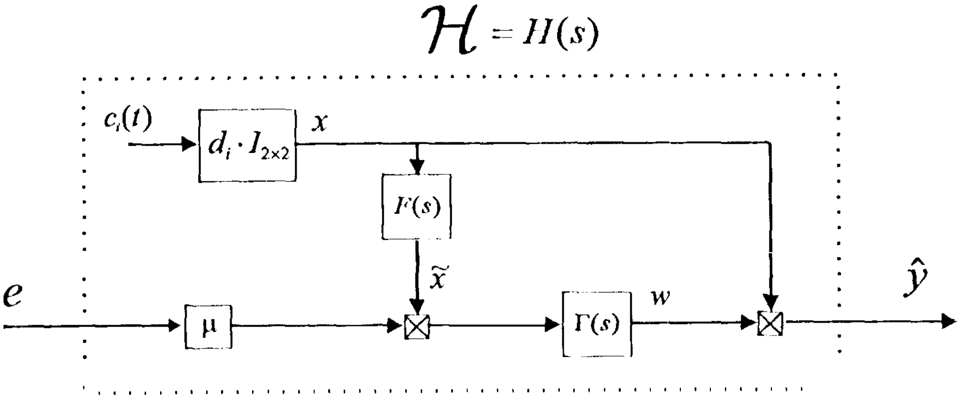


Figure 3.1: Equivalent LTI representation of a harmonic adaptive system with a single tone

LEMMA 3.1 (Single-Tone Regressor) *Let the regressor $x(t)$ in the harmonic adaptive system (2.1)-(2.5) be given by the single-tone expression,*

$$x(t) = d_i c_i(t) \tag{3.1}$$

where d_i is any scalar, and

$$c_i(t) = [\sin \omega_i t, \cos \omega_i t]^T \in R^2 \tag{3.2}$$

Then the mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly representable as the linear time-invariant operator,

$$\mathcal{H} \ : \ \hat{y} = H(p)e \tag{3.3}$$

where,

$$H(s) = \mu \ d_i^{\,2} \cdot H_i(s) \tag{3.4}$$

$$H_i(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) + \frac{F}{-}\,\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i) \tag{3.5}$$

$$F_R(i) \stackrel{\Delta}{=} Re(F(j\omega_i)); \quad F_I(i) \stackrel{\Delta}{=} Im(F(j\omega_i)) \tag{3.6}$$

Proof:

The filtered regressor (2.3) is composed of a single sinusoid at ω_i put through a linear filter $F(s)$. Hence, using (3.6) it can be written (in steady-state) as,

$$\hat{x}(t) = d_i F(p) c_i(t) = d_i \mathcal{F}_i \, c_i(t) \tag{3.7}$$

where,

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2} \tag{3.8}$$

Using (3.7), the mapping from e to $\hat{y}$ can be written as,

$$\hat{y} = \mu d_i^{\,2} c_i(t)^T \cdot \Gamma(p)\Big[\mathcal{F}_i c_i(t) e\Big] \tag{3.9}$$

Let $\gamma(t)$ be the impulse response of the LTI operator corresponding to $\Gamma(p)$. Then using (3.2) and (3.8), equation (3.9) can be expressed in terms of convolution integrals,

$$\begin{aligned} \hat{y} &= \mu d_i^{\,2} \sin\omega_i t \int_0^t \gamma(\tau)\Big[F_R(i)\sin\omega_i(t-\tau) + F_I(i)\cos\omega_i(t-\tau)\Big] e(t-\tau)\,d\tau \\ &+ \mu d_i^{\,2} \cos\omega_i t \int_0^t \gamma(\tau)\Big[F_R(i)\cos\omega_i(t-\tau) - F_I(i)\sin\omega_i(t-\tau)\Big] e(t-\tau)\,d\tau \qquad (3.10) \\ &= \mu d_i^{\,2} \int_0^t \gamma(\tau)\Big[F_R(i)\cos\omega_i\tau + F_I(i)\sin\omega_i\tau\Big] e(t-\tau)\,d\tau \qquad (3.11) \end{aligned}$$

Here, **(3.11)** follows from **(3.10)** by using standard trigonometric identities (see Remark 3.1 below). Note that (3.11) is in the form of a convolution of the input $e(t)$ with the *time-invariant* impulse response,

$$\tilde{\gamma}(t) \triangleq \gamma(t)\Big[F_R(i)\cos\omega_i t + F_I(i)\sin\omega_i t\Big] \tag{3.12}$$

Taking the Laplace transform $\mathcal{L}\{\cdot\}$ of (3.12) and using the modulation property [3],

$$\mathcal{L}\{\gamma(t)e^{j\omega_i t}\} = \Gamma(s - j\omega_i) \tag{3.13}$$

gives the desired expression (3.5). ❑

REMARK 3.1 In the proof of Lemma 3.1, (3.11) follows from (3.10) by using trigonometric identity,

$$\begin{aligned} &\sin\omega_i t\Big[F_R(i)\sin\omega_i(t-\tau) + F_I(i)\cos\omega_i(t-\tau)\Big] \\ &+ \cos\omega_i t\Big[F_R(i)\cos\omega_i(t-\tau) - F_I(i)\sin\omega_i(t-\tau)\Big] \\ &= F_R(i)\cos\omega_i\tau + F_I(i)\sin\omega_i\tau \qquad (3.14) \end{aligned}$$

Identity (3.14) is a slight generalization of (2.8), and shows that the function of both t and τ on the left-halld side can be represented purely as the function of τ on the right side. This ensures that the convolution (3.1 1) has a shift-invariant kernel, which indicates that the operator from e to $\hat{y}$ is 1:1'1. ●

3.2 Multitone Regressor Case

The main result of this paper is given next which gives necessary and sufficient conditions for the operator $\mathcal{H}$ to be LTI in the general multitone case.

THEOREM 3.1 (LTI Representation Theorem) *Let the regressor $x(t)$ in the harmonic adaptive system* (2.1)-(2.3) *be given by the general multitone harmonic expression (2.4)(2.5) where the frequencies $0 < \omega_1 < \ldots < \omega_m$ are distinct, Ilon.zero, and $|F(j\omega_i)| > 0$ for all i.*

Then,

***(i)** The mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly represent able as the linear time-inva riant operator,*

$$\mathcal{H}: \quad \hat{y} = H(p)e \tag{3.15}$$

if and only if the matrix $\mathcal{X}$ in (2.4) satisfies th e following X- Orthogon ality (XO) condition,

A'- *Orthogonality (XO) Condition:*

$$\mathcal{X}^T\mathcal{X} = D^2 \tag{3.16}$$

$$D^2 \triangleq \begin{bmatrix} d_1^2 \cdot I_{2\times 2} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & d_m^2 \cdot I_{2\times 2} \end{bmatrix} \in R^{2m\times 2m} \tag{3.17}$$

wh ere, $d_i^2 \geq 0$, $i = 1, \ldots, m$ are scalars *and $I_{2\times 2} \in R^{2\times 2}$ is the matrix identity.*

***(ii)** $H(s)$ in (3.15)* is given *in closed-form* as,

$$H(s) = \mu \sum_{i=1}^{m} d_i^2 \cdot H_i(s) \tag{3.18}$$

$$H_i(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) - \frac{F_I(i)}{2j}\Big(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i)\Big) \tag{3.19}$$

$$F_R(i) \triangleq Re(F(j\omega_i)); \; F_I(i) \triangleq Im(F(j\omega_i)) \tag{3.20}$$

Proof: See Appendix A. ■

Intuitively, the results of Theorem 3.1 can be understood using the sequence of block diagram rearrangements shown in Figure 3.2, (which incidentally can be taken as an alternative proof of sufficiency, but not necessity). Specifically, Figure 3.2 Part a. shows the initial adaptive system with harmonic regressor; Part b. Snows the matrix $\mathcal{X}$ pushed through several scalar matrix blocks of the diagram; Part c. substitutes the identity $\mathcal{X}^T\mathcal{X} = D^2$ where D^2 has the special pairwise diagonal fern] associated with the XO condition (3.16)(3.1'7); Part d. pushes the matrix D^2 back through several scalar matrix blocks:, and Part e. follows by recognizing that Part d. is simply a parallel bank of filters of the form shown in Figure 3.1 each with a perfect sille/cosine basis, i.e., it is representable as a summation of LTI systems of the form treated in Lemma 3.1.

DEFINITION 3.1 *The matrix $\mathcal{X}^T\mathcal{X} = D^2$ having the special pairwise diagonal structure (3.17) in Theorem 3.1 is defined as the* **confluence matrix** *associated with a particular harmonic adaptive system (2.1)-(2.5).* ■

The name "confluence matrix" has been chosen to reflect the fact that N signal channels are effectively combined into a smaller number of $2m$ channels in Figure 3.2 using properties of this matrix.

REMARK 3.2 The LTI representation from e to $\hat{y}$ in Theorem 3.1 is invariant under any orthogonal transformation of the regressor, i.e., any $z = Qx$ where $QQ^T = Q^TQ = I$. To see this, assume that $\mathcal{X}^T\mathcal{X} = D^2$, and denote $\mathcal{X}_z = Q\mathcal{X}$. Then using regressor z in the transformed system gives,

$$\mathcal{X}_z^T\mathcal{X}_z = \mathcal{X}^TQ^TQ\mathcal{X} = \mathcal{X}^T\mathcal{X} = D^2 \tag{3.21}$$

which satisfies the XO condi ion with the same confluence matrix D^2 as the original system. ■

REMARK 3.3 A harmonic adaptive system which does not satisfy the XO condition for a specific $\mathcal{X}$ can be made I,"J'] (assuming that $\mathcal{X}^T\mathcal{X}$ is invertible) by the regressor transformation $z = Rx$ where $R = D(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T$, and where D is any matrix chosen such that D^2 has the pairwise diagonal form (3.17), Then testing the XO condition for the transformed regressor z gives,

$$\mathcal{X}_z^T\mathcal{X}_z = \mathcal{X}^TR^TR\mathcal{X} = D^2 \tag{3.22}$$

which is satisfied by construction with confluence matrix D^2. ■

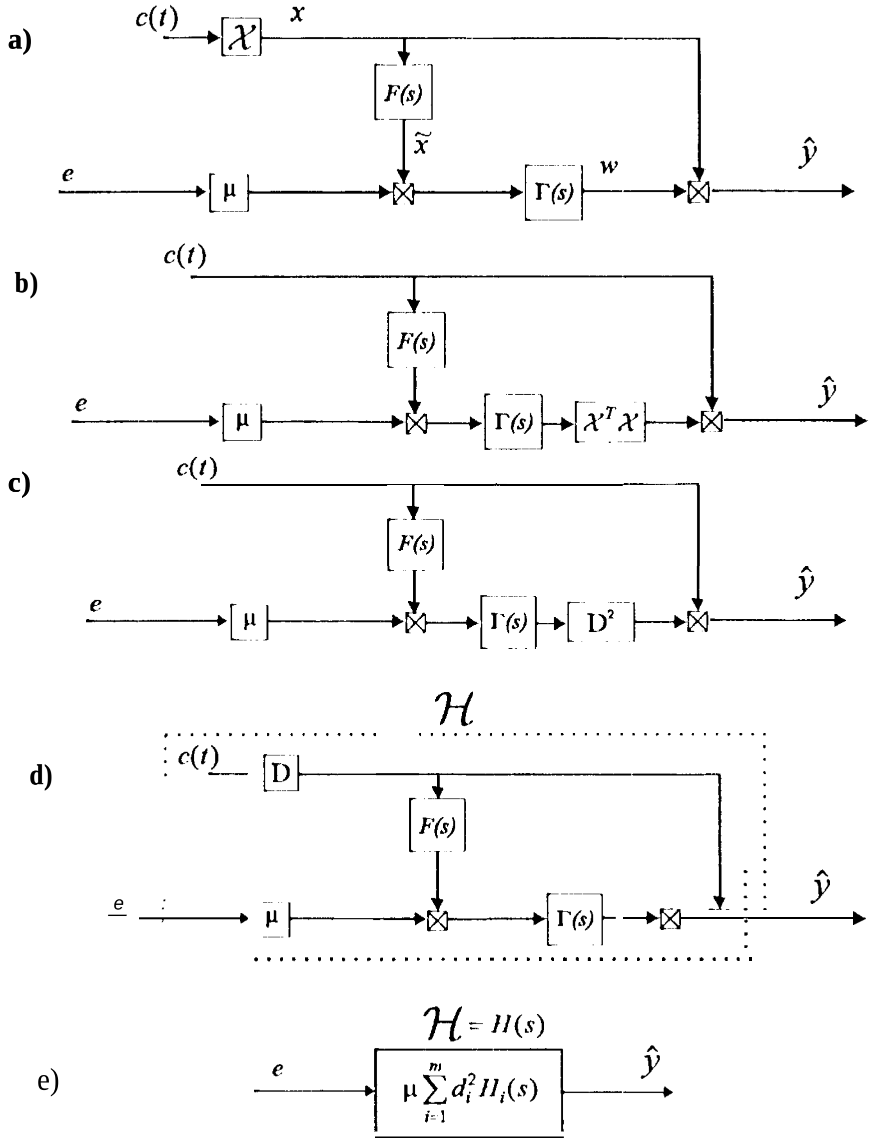


Figure 3.2: The X() condition of Theorem 3.1 motivated by sequence of Block diagram rearrangements

3.3 Minimal Realizations

Without loss of generality, the confluence matrix will be assumed to be nonsingular, i.e., $D^2 > 0$, since any zero diagonal pair $d_i^2 \cdot I_{2\times 2}, d_i^2 = 0$ in $D^2 \geq 0$ corresponds to a distinct frequency ω_i which can be removed from the definition of $c(t)$, reducing the Value of m accordingly.

Consider an LTI harmonic adaptive system defined by the regressor $x = \mathcal{X}c(t) \in R^N$, $\mathcal{X} \in R^{N\times 2m}$ having the confluence matrix $\mathcal{X}^T\mathcal{X} = D^2 > 0$. Let a different regressor be defined by $x_1 = \mathcal{X}_1 c(t)$, where $\mathcal{X}_1 \in R^n$ is any matrix factor of the same confluence matrix, i.e.,

$$\mathcal{X}_1^T\mathcal{X}_1 = D^2 \tag{3.23}$$

Then by the results of Theorem 3.1 the regressor x_1 will give an *equivalent realization* of the adaptive system in the sense that it will have an identical LTI transfer function $\mathcal{H}$.

In words, *the set of all equivalent realizations of a given LTI harmonic adaptive system is one-to-one with the set of all matrix factors of its confluence matrix.* An important class of equivalent realizations is considered next.

DEFINITION 3.2 *A **minimal realization** of an LTI harmonic adaptive system (2.1)-(2.5) is defined by the regressor choice $x_1 = \mathcal{X}_1 c(t)$ where $\mathcal{X}_1 \in R^{2m\times 2m}$ is any square matrix factor of its confluence matrix $D^2 > 0$.* ■

Minimal realizations are not unique since there are generally many square matrix factors of the pairwise diagonal confluence matrix. Minimal realizations arc important because the corresponding regressor $x_1 = \mathcal{X}_1 c(t) \in R^{2m}$ is of length $2m$ which is the minimum possible for realizing the LTI transfer function $\mathcal{H}$ of order $2m$. The number of tap weights will also be minimal of size $2m$.

An important property of regressors is persistent excitation (PE). Specifically, a regressor x is said to be PE if there exists $\alpha_1, \alpha_2, \delta > 0$ such that [26],

$$\alpha_2 \cdot I \geq \int_{t_0}^{t_0+\delta} x(\tau)x(\tau)^T d\tau \geq \alpha_2 \cdot I \qquad \text{for all } t_0 \geq 0 \tag{3.24}$$

Due to the sinusoidal structure of $c(t) \in R^{2m}$ in (2.5), the regressor $x = \mathcal{X}_1 c(t)$ associated with any minimal realization will be PE (i. e., m distinct sinusoidal frequencies are available for estimating $2m$ parameters). Consequently, *any LTI harmonic adaptive system (even overparametrized!) is input/output equivalent to a minimal realization with a PE regressor.* This is significant since the minimal realization can replace the original system in a Lyapunov type analysis to prove exponential (rather than just asymptotic) convergence and BIBO stability of a closed-lool) implementation. These strong properties are somewhat remarkable in light of the fact that the XO condition places no restriction on overparametrization of the adaptive system. However, it can be understood by the adaptive system's exact input/output equivalence to an LTI system, for which it is known that asymptotic stability implies exponential stability.

In a larger context, the property of input/output equivalence to a minimal PE realization is not restricted to adaptive systems with LTI representations. This same property has been shown to hold generically for overparametrized adaptive systems with bounded periodic regressors in Bayard, Spanos and Rahman [1].

3.4 Tonal Canonical Form

The reduced representation shown in Part d. of Figure 3.2 deserves special attention.

DEFINITION 3.3 ***Tonal*** *canonical* ***form*** *is defined as the unique minimal realization of an LTI harmonic adaptive system (2.1)-(2.5) specified by the regressor choice* $x_1 = \mathcal{X}_1 c(t)$ *where* $\mathcal{X}_1 \in R^{2m \times 2m}$ *is the unique positive diagonal square-root* $\mathcal{X}_1 = D > 0$ *of its confluence matrix* D^2. ●

Tonal canonical form corresponds to realizing the adaptive system with the simple paired sine/cosine regressor $x_1 = Dc(t)$. The name has been chosen to reflect the fact that each element of the regressor is a single pure tone. The realization is a canonical form in the sense that it is minimal length, always exists and is unique. Simply stated, *any harmonic adaptive system which admits an LTI representation is equivalent to an adaptive system realized in tonal canonical form i.e., with a minimal length paired sine/cosine regressor* $x_1 = Dc(t)$.

LTI adaptive systems arising from a paired sine/cosine regressor of the form $x_1 = Dc(t)$, have been studied by many researchers. A rigorous proof of its LTI properties was first given in Glover's 1977 paper (cf., [12], first paragraph of Section V) in the discrete-time case without regressor filtering (i. e., $F(s) = 1$), and for the gradient algorithm $P(s) = 1/s$. This result was extended later by Morgan and Sanford [20] to include an arbitrary regressor filter $F(s) \neq 1$, and recently by Collins [7] to include both a regressor filter and a general adaptation law. Presently the discrete-time version of this sine/cosine result is well-known in the signal processing community, and is included in the book by Widrow and Stearns (discrete-time case, [34], page 318).

Clearly, the paired sine/cosine regressor is well studied and has been used in adaptive systems for many years in the literature. A main point of this paper is that this is not just an isolated example of an adaptive system with an LTI representation, but rather is the fundamental canonical representation for all possible LTI harmonic adaptive systems.

4 SPECIAL CASES

Several useful LTI representations fall out as special cases of Theorem 3.1, and will be treated in the next few Corollaries.

COROLLARY 4.1 (Gradient Algorithm with Leakage) *Assume that the adaptive system with harmonic regressor (2.1)-(2.5) is specified as the gradient adaptive algorithm with leakage, i.e.,*

$$\dot{w} = -\sigma w + x(t)e(t) \tag{4.1}$$

for some value of the leakage parameter $\sigma \geq 0$ (cf., Ioannou and Kokotovic [15]). Then, if the XO condition of Theorem 3.1 is satisfied, the LTI expression (3.18) for $\bar{H}$ is given by,

$$\bar{H}(s) = \mu \sum_{i=1}^{m} {d_i}^2 \cdot \frac{s+\sigma}{s^2 + 2\sigma s + (\omega_i^2 + \sigma^2)} \tag{4.2}$$

PROOF: Result (4.2) follows by substituting, $\Gamma(s) = \frac{1}{s+\sigma}$, $\sigma \geq 0$, and $F(s) = 1$ in Theorem 3.1, and rearranging. ●

COROLLARY 4.2 (Gradient Algorithm) *Assume that the adaptive system with harmonic regressor (2.1)-(2.5) is specified as the gradient adaptive algorithm, i.e.,*

$$\dot{w} = \mu x(t)e(t) \tag{4.3}$$

Then, if the XO condition of Theorem 3.1 is satisfied, the LTI expression (3.18) for $\bar{H}$ is given by,

$$\bar{H}(s) = \mu \sum_{i=1}^{m} \frac{{d_i}^2 s}{s^2 + \omega_i^2} \tag{4.4}$$

PROOF: Result (4.4) follows by substituting $\sigma = 0$ into (4.2) of Corollary 4.1, and rearranging. ●

COROLLARY 4.3 (Filtered-X Algorithm) *Assume that the adaptive system with harmonic regressor (2.1)-(2.5) is specified as the Filtered-X algorithm (cf., [34]), using gradient adaptation, i.e.,*

$$\dot{w} = \mu \hat{x}(t)e(t) \tag{4.5}$$

$$\hat{x} = F(p)x(t) \tag{4.6}$$

for some choice of regressor filter $F(s)$.

Then, if the XO condition of Theorem 3.1 is satisfied, the LTI expression (3.18) for $\bar{H}$ is given by,

$$\bar{H}(s) = \hat{H}(s) \triangleq \mu \sum_{i=1}^{m} {d_i}^2 \cdot \frac{F_R(i)s + F_I(i)\omega_i}{s^2 + \omega_i^2} \tag{4.7}$$

PROOF: Result (4.7) follows by substituting $F(s) = \frac{1}{s}$ in Theorem 3.1, and rearranging. ■

COROLLARY 4.4 (Augmented Error Algorithm) *Assume that the adaptive system with harmonic regressor (2.1)-(2.5) is specified as the Augmented Error algorithm (cf., [19],[22]), using the gradient adaptation algorithm, i.e.,*

$$\dot{w} = \mu \hat{x}(t)\epsilon(t) \tag{4.8}$$

where the augmented error ϵ is given by,

$$\epsilon = e + F(p)[\hat{y}] - \tilde{y} \tag{4.9}$$

$$\hat{y} = w^T x \tag{4.10}$$

$$\tilde{y} = w^T \hat{x} \tag{4.11}$$

$$\hat{x} = F(p)[x] \tag{4.12}$$

for some choice of regressor filter $F(p)$.

Then, if the XO condition of Theorem 3.1 is satisfied, the mapping from e to $\hat{y}$ is LTI and is given by,

$$\overline{H}(s) = \hat{H}(s)(1 + \hat{C}(s) - F(s)\hat{H}(s))^{-1} \tag{4.13}$$

where $\hat{H}(s)$ is defined in (4.7), and,

$$\hat{C}(s) = \mu \sum_{i=1}^{m} d_i^2 \cdot \frac{|F(j\omega_i)|^2 s}{s^2 + \omega_i^2} \tag{4.14}$$

PROOF: Using (4.8) and (4.10) together, the mapping from ϵ to $\hat{y}$ can be simply recognized as the Filtered-X algorithm with filter $F(s)$ and can be calculated with the aid of Corollary 4.3 to give,

$$\hat{y} = \hat{H}(p)\epsilon \tag{4.15}$$

where $\hat{H}(s)$ is given by (4.7). Similarly, using (4.8) and (4.11) together, the mapping from ϵ to $\tilde{y}$ is of the form of a gradient algorithm with regressor $\hat{x} = \mathcal{X}\mathcal{F}c(t)$, where $\mathcal{F}$ is defined as,

$$\mathcal{F} \triangleq blockdiag\{\mathcal{F}_i\} \in R^{2m \times 2m} \tag{4.16}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i = 1, ..., m \tag{4.17}$$

The mapping from ϵ to $\tilde{y}$ can be calculated with the aid of (4.4) in Corollary 4.2 to give,

$$\tilde{y} = \hat{C}(p)\epsilon \tag{4.18}$$

where $\hat{C}(s)$ is given by (4.14), since the related XO condition is satisfied with,

$$\mathcal{F}^T \mathcal{X}^T \mathcal{X} \mathcal{F} = blockdiag\{d_i{}^2 \cdot |F(j\omega_i)|^2 I_{2\times 2}\} \tag{4.19}$$

Substituting (4.15) and (4.18) into (4.19) gives upon rearranging,

$$e = \left(1 + \hat{C}(p) - F(p)\hat{H}(p)\right)^{-1} e \tag{4.20}$$

Substituting (4.20) into (4.15) gives the desired result (4.13). ■

5 APPLICATION

For demonstration purposes, the theory of LTI representations developing in this paper is applied to the problem of harmonic noise cancellation. The problem of cancelling harmonic noise arises in many diverse fields. Applications include damping vibrations in flexible structures *[28] [30],* helicopters [13][27], propeller aircraft [18][8][9], air conditioning ducts [14][6], automobile engines [24][25], cryocoolers [7], rotating machinery [29] [31], submarines, or acoustic. noise control [23], and electrical noise [34], etc..

A general formulation of the harmonic suppression problem is shown in Figure 5.1.

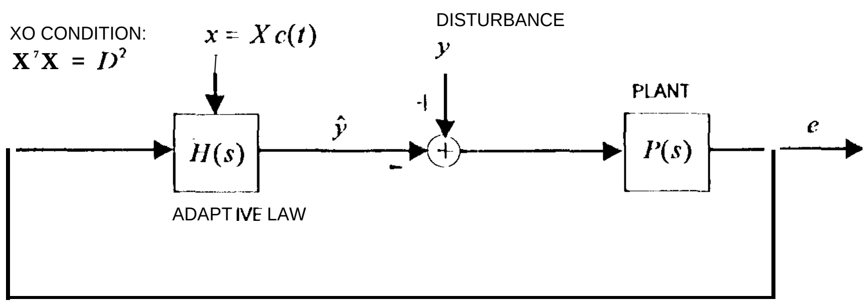


Figure 5.1: Harmonic noise suppression problem

Assuming that the adaptive law admits an LTI representation *H(s),* the closed-loop transfer function from the y to e in Figure 5.1 can be calculated as,

$$E(s) = -\frac{P(s)}{1 + H(s)P(s)} Y(s) \tag{5.1}$$

where $E(s)$ and $Y(s)$ denote the Laplace transformed signals e and y, respectively.

5.1 Gradient Algorithm

Let the Gradient algorithm of Corollary 4.2 be used for adaptation, and consider the special Case where $P(s) = 1$. Then the closed-loop LTI system becomes,

$$E(s) = \frac{1}{1 + H(s)} \cdot Y(s) \tag{5.2}$$

where,

$$H(s) = \mu \sum_{i=1}^{m} \frac{d_i^2 s}{s^2 + \omega_i^2} \tag{5.3}$$

Equivalently,

$$E(s) = \frac{\prod_{i=1}^{m} (s^2 + \omega_i^2)}{\prod_{i=1}^{m} (s^2 + \omega_i^2) + \mu s \sum_{i=1}^{m} d_i^2 \prod_{j \neq i} (s^2 + \omega_j^2)} \cdot Y(s) \tag{5.4}$$

From the numerator in (5.4), it is seen that there are "perfect notches" at the frequencies $\{\omega_i\}_{i=1}^{m}$. Using root locus, the pole locations can be found analytically for small μ as [2],

$$s_i^{\pm} = -\frac{\mu d_i^2}{2} \pm j\omega_i, \; i = 1, \ldots m \tag{5.5}$$

As μ increases the roots move in a direction perpendicular to the $j\omega$ axis directly into the left half plane, a distance of $\mu d_i^2/2$. Hence, the resulting damping in the ith disturbance tone is proportional to both the adaptation gain μ and the diagonal entry of the confluence matrix d_i^2. From the pole and zero locations it can be determined that the notches are symmetric about each frequency ω_i with 3dB bandwidth μd_i^2 in radians/sec.

Since the loop gain (5.3) is completely in the RHP, a Nyquist analysis indicates that the closed-loop system is completely phase stabilized (i.e., there is no gain crossover frequency) and will be stable for any values of $\mu, \omega_i, d_i^2 \geq 0$, $i = 1, \ldots, m$.

5.2 Filtered-X (FX) Algorithm

The Filtered-X algorithm is a general method to deal with the intervening plant $P(s)$ in the error path [34]. The Filtered-X algorithm first appeared in a paper by Widrow 1971 [32] treating the special case of a pure delay plant. Later extensions to arbitrary LTI plants appeared at approximately the same time in papers by Morgan 1980 [21], Burgess 1981 [5], and Widrow et. al. [33].

Let the Filtered-X algorithm of Corollary 4.3, be used for adaptation with the filter choice $F(s) = \hat{P}(s)$ where $\hat{P}(s)$ is an estimate of $P(s)$. Then the closed-loop system is given by,

$$E(s) = \frac{P(s)}{1 + \hat{H}(s) P(s)} \cdot Y(s) \tag{5.6}$$

where,

$$\hat{H}(s) = \mu \sum_{i=1}^{m} \frac{d_i^2 \left(\hat{P}_R(i) s + \hat{P}_I(i) \omega_i \right)}{s^2 + \omega_i^2} \tag{5.7}$$

$$\hat{P}_R(i) \triangleq Re(\hat{P}(j\omega_i)); \quad \hat{P}_I(i) \triangleq Im(\hat{P}(j\omega_i)) \tag{5.8}$$

Specifying the plant as a rational function $P(s) = N(s)/D(s)$ in (5.6) and rearranging gives the closed-loop system as,

$$E(s) = \frac{N(s)\prod_{i=1}^{m}(s^2+\omega_i^2)}{D(s)\prod_{i=1}^{m}(s^2+\omega_i^2)+\mu N(s)\sum_{i=1}^{m} d_i^{\,2}\left(\hat{P}_R(i)s+\hat{P}_I(i)\omega_i\right)\prod_{j\neq i}(s^2+\omega_j^2)} \cdot V(s) \tag{5.9}$$

From the numerator in (5.9), it is seen that there are "perfect notches" at the frequencies $\{\omega_i\}_{i=1}^{m}$. Using root locus (assuming that $P(s)$ does not have resonances near the resonances of $\hat{H}(s)$) the pole locations can be calculated for small μ as,

$$s_i^{\pm} = -\frac{\mu d_i^{\,2}}{2} \cdot \hat{P}^*(j\omega_i)P(j\omega_i) \pm j\omega_i \tag{5.10}$$

It is seen that for small μ the pole moves off the $j\omega$ axis a distance proportional to $\frac{\mu d_i^{\,2}}{2}\left|\hat{P}^*(j\omega_i)P(j\omega_i)\right|$, at a departure angle determined by the phase of $\hat{P}^*(j\omega_i)P(j\omega_i)$. Hence, the estimate $\hat{P}(s)$ must approximate the plant $P(s)$ to within $\pm 90°$ in the vicinity of each resonance frequency ω_i, $i = 1, \ldots, m$ as a necessary condition for stability.

Stability conditions can be understood more clearly using a Nyquist analysis. For this purpose, the loop gain in (5.6) is rewritten as,

$$\hat{H}(s)P(s) = \mu\sum_{i=1}^{m} \frac{d_i^{\,2}\left(\hat{P}_R(i)s+\hat{P}_I(i)\omega_i\right)P(s)}{s^2+\omega_i^2} \tag{5.11}$$

$$= \mu\sum_{i=1}^{m} \frac{d_i^{\,2}s}{s^2+\omega_i^2} \cdot L_i(s) \cdot P(s) \tag{5.12}$$

where $L_i(s) = \hat{P}_R(i) + (\omega_i/s)\hat{P}_I(i)$. Evaluating $L_i(s)$ at the resonance $s = j\omega_i$ gives,

$$L_i(j\omega_i) = \hat{P}_R(i) - j\hat{P}_I(i) = \hat{P}^*(j\omega_i) \tag{5.13}$$

where the superscript "$*$" denotes the complex conjugate. It is noted that $L_i(s)$ *acts to approximate the complex conjugate of the plant at the resonance frequency* ω_i. Comparing the loop gain (5.12) with (5.3), and using (5.13) it is seen that the Filtered-X algorithm works by trying to "conjugate" the plant phase in the vicinity of each resonance frequency ω_i, in an attempt to recover the RHP loop gain (and hence the convergence properties) associated with the gradient algorithm applied to a unit plant $P(s) = 1$. This corresponds to phase stabilization on the Nyquist plot in the vicinity of each of the resonances. Away from the resonances, the loop gain (5.12) can be gain stabilized by making the adaptation gain μ sufficiently small to avoid the Nyquist critical point. (This provides a precise condition on μ which avoids the heuristic "sufficiently slow tuning to justify commuting LTV blocks" argument used in other treatments [34]). Hence, the Filtered-X algorithm works by using a combination of phase and gain stabilization. For any specific example, precise stability bounds can be established on all parameters $\mu, \omega_i, d_i^{\,2} \geq 0$, $i = 1, \ldots, m$ and the error $P(s) - \hat{P}(s)$ using a more detailed LTI analysis.

6 DISCUSSION

At this point, several comments are in order.

1. All of the LTI transfer functions $\tilde{H}(s)$ in the Corollaries of Section 4 have large gains in the vicinity of the tone frequencies ω_i, $i = 1, \ldots, m$. When used in closed-loop, the large gains become "notches" of the form $(1 + \tilde{H}(s))^{-1}$ (cf., Section 5). The creation of closed-loop notches is simply an application of the internal model principle (IMP) [11] which has also been discussed in the context of adaptive feedforward systems by Bodson *et. al.* [4]. These closed-loop notches are effective at cancelling sinusoidal disturbances, and have been used for this purpose in a wide variety of adaptive feedforward control applications (cf., Sievers and von Flotow [28], Morgan [21], Collins [7], Spanos and Rahman [30], Bodson, Sacks and Khosla [4], and Messner and Bodson [17]).

2. The LTI result, in Corollary 4.4 for the Augmented Error algorithm appears to be new and does not have any counterpart in the literature. This result has been shown in [2] to provide an important alternative strategy for adaptive harmonic noise cancellation where there is a plant blocking the noise cancellation path. In contrast to the Filtered-X algorithm which attempts to phase stabilize near the resonances and gain stabilize elsewhere, the Augmented Error algorithm attempts to phase stabilize everywhere [2].

3. Glover's result [12] on LTI representations of adaptive systems with long Lal)-delay line (TDL) regressors can also be shown to be a special case of the XO condition [2]. The rigorous extension of Glover's result to the multitone case has been made in [2] based on the XO condition.

4. If the XO condition is not satisfied exactly, the adaptive system is no longer LTI. However, it is instead representable as an LTI and LTV subsystem in parallel, where the LTV part can be explicitly norm-bounded [2]. The norm-bound (an induced 2-norm) is compatible with standard H_∞ robustness analysis.

7 CONCLUSIONS

This paper establishes a necessary and sufficient condition for a harmonic adaptive system to admit an exact LTI representation. This condition (i.e., the XO condition) unifies many results in the literature, and leads to the notion of equivalent realizations, minimal realizations, and the tonal canonical form. Simply stated, the XO condition indicates that the block diagram of the adaptive system can be rearranged so that the regressor has a paired sine/cosine form (i.e., tonal canonical form). These results are important because LTI adaptive systems can be designed and analyzed completely using standard methods, taking advantage of a wealth of tools available for LTI systems.

Minimal realizations are important since their corresponding regressor is persistently exciting, and is of minimum length for realizing the resulting LTI transfer function. The tonal canonical realization is important because it indicates equivalence to an adaptive system realized with a minimal length paid sine/c.osjnc regressor. The paired sine/cxxsillc regressor is w c]] studied and has been used in adaptive systems for many years in the literature. It is interesting and significant that it turns out to be the canonical representation of all harmonic adaptive systems with LTI representations.

Regressors for adaptive sinusoidal noise cancellation are typically constructed by filtering and combining various signals measured in the environment. The XO condition allows a systematic comparison of the quality of such implementations, and is expected to lead to many optimized basis functions and new adaptive control architectures in the future.

8 ACKNOWLEDGEMENTS

Acknowledgements are due to the J 1'1, Interferometer Technology 1 'rogram under the direction of Dr. R. Laskin for supporting this research. The author would like to thank Dr. J. Spanos and Dr. Z. Rahman of the JPL, and Prof. G. Papavassilop oulos of USC for several relevant technical discussions. This research was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

A APPENDIX : Proof of Theorem 3.1

Define,

$$\mathcal{X}^T\mathcal{X} \triangleq \mathrm{M} = \{\mathrm{M}_{ij}\} \in R^{2m\times 2m} \tag{A.1}$$

$$\mathrm{M}_{ij} \triangleq \begin{bmatrix} \mathrm{m}_{ij}^{11} & \mathrm{m}_{ij}^{12} \\ \mathrm{m}_{ij}^{21} & \mathrm{m}_{ij}^{22} \end{bmatrix} \in R^{2\times 2}; \quad \text{for } i,j = 1,...,m \tag{A.2}$$

Using (2.3)-(2.5), the filtered regressor can be represented as,

$$\hat{x} = F(p)[x] = F(p)[\mathcal{X}c(t)] = \mathcal{X}\mathcal{F}c(t) \tag{A.3}$$

where $\mathcal{F}$ is the block diagonal matrix given by,

$$\mathcal{F} = blockdiag\{\mathcal{F}_i\} \in R^{2m\times 2m} \tag{A.4}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i = 1,...,m \tag{A.5}$$

Proof of (i): It is desired to snow that $M = D^2$ (where D^2 has the block-diagcma] form (3.17)), if and only if the mapping $\mathcal{H}$ from e to $\hat{y}$ is LTI. From (2.1)-(2.5) and (A.3) this mapping can be written as,

$$\hat{y} = \mu c(t)^T \mathcal{X}'^T \cdot \Gamma(p)[\mathcal{X}'\mathcal{F}c(t)e] \tag{A.6}$$

$$= \mu c(t)^T \mathcal{X}'^T \mathcal{X}'\mathcal{F} \cdot \Gamma(p)[c(t)e] \tag{A.7}$$

$$= \mu c(t)^T \mathrm{M}\mathcal{F} \int_0^t \gamma(\tau)c(t-\tau)e(t-\tau)d\tau \tag{A.8}$$

$$= \mu \int_0^t \gamma(\tau)c(t)^T\mathcal{V}c(t-\tau)e(t-\tau)d\tau \tag{A.9}$$

where $\gamma(t)$ is the impulse response of the filter $\Gamma(s)$, and where we have defined the matrix,

$$\mathcal{V} = \mathrm{M}\,\mathcal{F} \tag{A.10}$$

For later convenience, V is partitioned into 2 **x** 2 blocks (compatibly with $\mathcal{F}$,M), as follows,

$$\mathcal{V} = \{\mathcal{V}_{ij}\} \in R^{2m \times 2m} \tag{A.11}$$

$$\mathcal{V}_{ij} \triangleq \begin{bmatrix} v_{ij}^{11} & v_{ij}^{12} \\ v_{ij}^{21} & v_{ij}^{22} \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i,j = 1,\ldots,m \tag{A.12}$$

It is seen that the mapping $\mathcal{H}$ from e to $\hat{y}$ in (A.9) is represented by a convolution integral, which is time-invariant if and only if the kernel is independent of time t, equivalently, if and only if,

$$c(t)^T\mathcal{V}c(t-\tau) = \beta(\tau) \tag{A.13}$$

where $\beta(\tau)$ is *a function purely of* τ. Condition (A.13) will be examined in detail. Expanding $c(t-\tau)$ gives the identity,

$$c(t-\tau) = Q(t)c(\tau) \tag{A.14}$$

where $Q(t)$ is the block diagonal matrix,

$$Q(t) = blockdiag\{Q_i(t)\} \in R^{2m\times 2m} \tag{A.15}$$

$$Q_i(t) \triangleq \begin{bmatrix} -\cos\omega_i t & \sin\omega_i t \\ \sin\omega_i t & \cos\omega_i t \end{bmatrix} \in R^{2\times 2}; \qquad \text{for } i = 1,\ldots,m \tag{A.16}$$

Substituting (A.14) into (A.13) gives,

$$\alpha^T(t)c(\tau) = \beta(\tau) \tag{A.17}$$

where,

$$\alpha^T(t) \triangleq c(t)^T\mathcal{V}Q(t) \tag{A.18}$$

Equation (A.17) holds if and only if α is a constant vector, i.e., $\alpha(t) = \alpha^{\circ}$. To see this, multiply both sides of (A.17) on the right by $c^T(\tau)$ and integrate with respect to τ over any interval $[\tau_1, \tau_2]$ such that $\int c(\tau)c(\tau)^T d\tau$ is invertible. Such an interval always exists since the components of $c(\tau)$ are linearly independent functions (i.e., sines and cosines of distinct frequencies). The resulting equation can be solved for α, implying that any valid solution α to equation (A.17) must be a constant vector.

Assuming that α is constant, consider relation (A.18) taken two components at a time, i.e.,

$$c_i(t)^T \mathcal{V}_{ij} Q_j(t) = [\alpha_1^{\circ}, \alpha_2^{\circ}] \tag{A.19}$$

where $\alpha_1^{\circ}, \alpha_2^{\circ}$ are constants and,

$$c_i(t) = [\sin \omega_i t, \cos \omega_i t]^T \tag{A.20}$$

Expanding the first component of (A.19) gives,

$$\begin{aligned} \alpha_1^{\circ} &= -\cos(\omega_j t)\sin(\omega_i t)v_{ij}^{11} - \cos(\omega_j t)\cos(\omega_i t)v_{ij}^{21} \\ &\quad + \sin(\omega_j t)\sin(\omega_i t)v_{ij}^{12} + \sin(\omega_j t)\cos(\omega_i t)v_{ij}^{22} \end{aligned} \tag{A.21}$$

$$\begin{aligned} &= \frac{1}{2}(-\sin(\omega_j + \omega_i)t + \sin(\omega_j - \omega_i)t)v_{ij}^{11} - \frac{1}{2}(\cos(\omega_j - \omega_i)t + \cos(\omega_j + \omega_i)t)v_{ij}^{21} \\ &\quad + \frac{1}{2}(\cos(\omega_j - \omega_i)t - \cos(\omega_j + \omega_i)t)v_{ij}^{12} + \frac{1}{2}(\sin(\omega_j + \omega_i)t + \sin(\omega_j - \omega_i)t)v_{ij}^{22} \end{aligned} \tag{A.22}$$

$$\begin{aligned} &= \frac{1}{2}\left[(v_{ij}^{22} - v_{ij}^{11})^2 + (v_{ij}^{21} + v_{ij}^{12})^2\right]^{\frac{1}{2}} \sin((\omega_j + \omega_i)t + \phi_{ij}) \\ &\quad + \frac{1}{2}\left[(v_{ij}^{11} + v_{ij}^{22})^2 + (v_{ij}^{12} - v_{ij}^{21})^2\right]^{\frac{1}{2}} \sin((\omega_j - \omega_i)t + \psi_{ij}) \end{aligned} \tag{A.23}$$

Here, (A.22) follows by expanding (A.21) in terms of sum/difference frequencies; and (A.23) follows by rearrangement. The constant phases ϕ_{ij}, ψ_{ij} can also be calculated, but will not be needed. A similar expression to (A.23) can be calculated by using the second term α_2° in (A.19), but this can be shown to be redundant with (A.23) and will not impose additional constraints.

Case 1: $i \neq j$

First consider the case where $i \neq j$ so that ω_i and ω_j are distinct nonzero frequencies. Then (A.23) is the sum of two sinusoids of distinct frequencies, which is equal to a constant if and only if both terms vanish identically, i.e.,

$$v_{ij}^{11} = v_{ij}^{22}; \quad v_{ij}^{21} = -v_{ij}^{12} \tag{A.24}$$

$$v_{ij}^{11} = -v_{ij}^{22}; \quad v_{ij}^{12} = v_{ij}^{21} \tag{A.25}$$

Equivalently, $v_{ij}^{11} = v_{ij}^{22} = v_{ij}^{21} = v_{ij}^{12} = 0$, which gives,

$$\mathcal{V}_{ij} = 0; \quad \text{for } i \neq j \tag{A.26}$$

However, from (A.10) and the 2 x 2 partitioned structure of matrices M and $\mathcal{F}$,

$$\mathcal{V}_{ij} = \mathrm{M}_{ij}\mathcal{F}_j \tag{A.27}$$

where $\mathcal{F}_j$ in (A.5) is invertible (since its determinant $|F(j\omega_j)|^2$ is nonzero by assumption). Combining (A.26) and (A.27), and using the invertibility of $\mathcal{F}_j$ gives,

$$\mathrm{M}_{ij} = 0; \quad \text{for } i \neq j \tag{A.28}$$

Case 2: $i = j$

Next consider the case where $i = j$. Then, equation (A.23) becomes,

$$\begin{aligned} \alpha_1^\circ = \; & \frac{1}{2}\left[(v_{ii}^{22} - v_{ii}^{11})^2 + (v_{ii}^{21} + v_{ii}^{12})^2\right]^{\frac{1}{2}} \sin(2\omega_i t + \phi_{ii}) \\ & + \frac{1}{2}\left[(v_{ii}^{11} + v_{ii}^{22})^2 + (v_{ii}^{12} - v_{ii}^{21})^2\right]^{\frac{1}{2}} \sin(\psi_{ii}) \end{aligned} \tag{A.29}$$

The second term of (A.29) is constant, as desired. The first term of (A.29) is sinusoidal of nonzero frequency, which is constant-valued if and only if it vanishes identically, i.e.,

$$v_{ii}^{11} = v_{ii}^{22}; \quad v_{ii}^{21} = -v_{ii}^{12} \tag{A.30}$$

However,

$$\mathrm{M}_{ii}\mathcal{F}_i = \mathcal{V}_{ii} \tag{A.31}$$

or equivalently (by the invertibility of $\mathcal{F}_i$),

$$\mathrm{M}_{ii} = \mathcal{V}_{ii}\mathcal{F}_i^{-1} \tag{A.32}$$

By the symmetry and nonegativity of $\mathrm{M} = \mathcal{X}^T\mathcal{X}$ one has,

$$\mathrm{m}_{ii}^{21} = \mathrm{m}_{ii}^{12} \tag{A.33}$$

$$\mathrm{m}_{ii}^{11} \geq 0; \quad \mathrm{m}_{ii}^{22} \geq 0 \tag{A.34}$$

Expanding (A.32) using properties (A.30)(A.33) and an analytic expression for $\mathcal{F}_i^{-1}$ gives,

$$\begin{bmatrix} \mathrm{m}_{ii}^{11} & \mathrm{m}_{ii}^{12} \\ \mathrm{m}_{ii}^{12} & \mathrm{m}_{ii}^{22} \end{bmatrix} = \mathcal{V}_{ii}\mathcal{F}_i^{-1} = \begin{bmatrix} d_i^{\,2} & \delta_i \\ -\delta_i & d_i^{\,2} \end{bmatrix} \tag{A.35}$$

where,

$$d_i^2 \triangleq (v_{ii}^{11} F_R(i) + v_{ii}^{12} F_I(i))/|F(j\omega_i)|^2 \tag{A.36}$$

$$\delta_i \triangleq (-v_{ii}^{11} F_I(i) + v_{ii}^{12} F_R(i))/|F(j\omega_i)|^2 \tag{A.37}$$

By (A.33) and the special form of the right-hand side of (A.35), it follows that, $m_{ii}^{11} = m_{ii}^{22} \triangleq d_i^2 \geq 0$ and $m_{ij}^{21} = m_{ij}^{12} = 0$, which gives,

$$M_{ii} = \begin{bmatrix} d_i^2 & 0 \\ 0 & d_i^2 \end{bmatrix} \geq 0 \tag{A.38}$$

In summary, the kernel of the convolution (A.9) is a function purely of τ if and only if the i,jth block M_{ij} of the matrix M has the form (A.28) for $i \neq j$, and the form (A.38) for $i = j$. Equivalently, the linear operator $\mathcal{H}$ from c to $\hat{y}$ is time-invariant if and only if M has the block-diagonal form of D^2 in (3.17) of Theorem 3.1, which is the desired result.

Proof of (ii): Substituting $\mathcal{X}^T\mathcal{X} \triangleq M = D^2$ into (A.7) gives,

$$\hat{y} = \mu c(t)^T D^2 \, \Gamma(p) \, [\mathcal{F} c(t) \, c] \tag{A.39}$$

$$= \mu \sum_{i=1}^{m} d_i^2 c_i(t)^T \cdot \Gamma(p)[\mathcal{F}_i c_i(t) c] \tag{A.40}$$

$$= \mu \sum_{i=1}^{m} d_i^2 \cdot H_i(p) c \tag{A.41}$$

Here, (A.40) follows by the partitioned structure of $D2, \mathcal{F}, c(t)$; and (A.41) follows by applying Lemma 3.1 (e.g., compare to (3.9)), separately for each term in the sum (A.40). ■

References

[1] D.S. Bayard, J. Spanos and Z. Rahman, "Exponential convergence of the tracking error in adaptive systems without persistent excitation," IEEE Conf. Decision and Control, New Orleans, LA, December 1995; Expanded version in JPL Internal Document D-12877, August 1995.

[2] D.S. Bayard, *A General Theory of Linear Time Invariant Adaptive Feedforward Systems with Harmonic Regressors*. Jet Propulsion Laboratory, Internal Document JPL D-13556, April 12, 1996.

[3] W.H. Beyer, Editor, *CRC Standard Mathematical Tables*. 28th Edition, Boca Raton, Florida: CRC Press, Inc, 1987.

[4] M. Bodson , A. Sacks, and P. Khosla, "Harmonic generation in adaptive feedforward cancellation schemes," IEEE Trans. Automatic Control, vol. 39, no. 9, pp. 1939-1944, September 1994.

[5] J .C. Burgess, "Active sound control in a duct: A computer simulation," J. Acoust. Soc. Amer., vol. 70, pp. 715-726, Sept. 1981.

[6] G. Canevet, "Active sound absorption in an air conditioning duct," J. Sound and Vibration, vol. 58, pp. 333-345, 1978.

[7] S.A. Collins, *Multi-Axis Analog Adaptive Feedforward Cancellation of Cryocooler Vibration.* Ph.D. Thesis, SERC# 8-94, MIT Space Engineering Research Center, MIT, Cambridge Massachusetts, April 1994.

[8] S.J. Elliott, P. A. Nelson, I.M. Stothers, and C.C . Boucher, "Preliminary results of in-flight experiments on the active control of propeller-induced cabin noise," J. Sound and Vibration, vol. 128, pp. 355-357, 1989.

[9] S.J. Elliott, P. A. Nelson, I.M. Stothers, and C.C. Boucher, "In-flight experiments on the active control of propeller-induced cabin noise," J. Sound and Vibration, vol. 140, pp. 219-238, 1990.

[10] S.J. Elliott, I.M. Stothers, and P. A. Nelson, "A multiple error LMS algorithm and its application to the active control of sound and vibration," IEEE Trans. Acoustics, Speech and Signal Processing, vol. ASSP-35, no. 10, pp. 1423-1434, October 1987.

[11] B. Francis and M.W. Wonham, "The internal model principle of control theory," Automatica, vol. 12, no. 5, pp. 457-465, 1976.

[12] J.R. Glover, Jr., "Adaptive noise cancelling applied to sinusoidal interferences," IEEE Trans. Acoustics, Speech and Signal Processing, vol. 25, no. 6, 484-491, December 1977.

[13] S.R. Hall and N. M. Wereley, "Linear control issues in the higher harmonic control of helicopter vibrations," in Proc. 45th, Annual Forum Amer. Helicopter Soc., Boston, MA, 1989.

[14] H. Hamada, T. Miura, M. Takahashi, and Y. Oguri, "Adaptive noise control system in air-conditioning ducts," Proc. of Inter-Noise '88, Avignon, vol. 2, pp. 1017-1020, 1988.

[15] P.A. Ioannou and P.V. Kokotovic, *Adaptive Systems with Reduced Models.* New York: Springer-Verlag, 1983.

[16] G. Long, F. Ling, and J.G . Proakis, "The LMS algorithm with delayed coefficient adaptation," IEEE Trans. Acoustics, Speech, and Signal Proc., vol. 37, no. 9, pp. 1397-1405, September 1989.

[17] W. Messner and M. Bodson, "Design of adaptive feedforward algorithms using internal model equivalence," Int. J. Adaptive Control and Signal Processing, vol. 9, no. 2, pp. 199-212, March-April 1995.

[18] F.B. Metzger, "Strategies for reducing propeller aircraft cabin noise," Automotive Engineering, vol. 89, pp. 107-113, 1981.

[19] R.V. Monopoli, "Model reference adaptive control with an augmented error signal," IEEE Trans. Automatic Control, vol. 19, pp.474-484, 1974.

[20] D.R. Morgan, and C. Sanford, "Control theory approach to the stability and transient analysis of the Filtered-X LMS adaptive notch filter," IEEE Trans. Signal Processing, vol. 40, no. 9, pp. 2341-2345, September 1992.

[21] D.R. Morgan, "An analysis of multiple correlation cancellation loops with a filter in the auxiliary path," IEEE Trans. Acoustics, Speech and Signal Proc., vol. 28, no. 4, pp. 454-467, August 1980.

[22] K.S. Narendra and A.M. Annaswamy, *Stable Adaptive Systems.* Prentice-Hall, Englewood Cliffs, New Jersey, 1989.

[23] P.A. Nelson and S.J. Elliott, *Active Control of Sound.* New York: Academic Press, 1992.

[24] L.J. Oswald, "Reduction of diesel engine noise inside passenger compartments using active adaptive noise control," Proc. of Inter-Noise '84, pp. 483-488, Honolulu, Hawaii, 1984.

[25] D.C. Perry, S.J. Elliott, I.M. Stothers, and S.J. Oxley, "Adaptive noise cancellation for road vehicles," Proc. Inst. of Mechanical Engineers Conf. on Automotive Electronics, pp. 150-163, 1989.

[26] S. Sastry and M. Bodson, *Adaptive Contro: Stability, Convergence and Robustness.* Prentice Hall, Englewood Cliffs, N.J., 1989.

[27] J. Shaw and N. Albion, "Active control of the helicopter rotor for vibration reduction," J. Amer. Helicopter Soc., vol. 26, no. 3, 1981.

[28] L.A. Sievers and A.H. von Flotow, "Comparison and extensions of methods for cancellation of periodic noise," IEEE Trans. Signal Processing, vol. 40, no. 10, pp. 2377-2391, October 1992.

[29] R.A. Smith and G.B.B. Chaplin, "A comparison of some Essex algorithms for major industrial applications," Proc. Inter-Noise '83, Edinburgh, England, vol. 1, pp. 407-410, 1983.

[30] J. Spare and Z. Rahman, "Narrow-band control experiments in active vibration isolation," SPIE Proc., Vibration Monitoring and Control, paper 2264-(11, San Diego, July 1994.

[31] M.A. Swinbanks, "The active control of low frequency sound in a gas turbine compressor installation," Proc. Inter-Noise '82, San Francisco, vol. 2, pp. 423-426, 1982.

[32] B. Widrow, "Adaptive filters," in *Aspects of Network and System Theory,*, R.E. Kalman and N. De Claris, Eds., New York: Holt Rinehart, Winston 1971.

[33] B. Widrow, D. Shur, and S. Shaffer, "On adaptive inverse control," in Proc. 15th Asilomar Conf. Circuits, Systems. Computation, PP. 185-189. 1981.

[34] B. Widrow and S.D. Stearns, *Adaptive Signal Processing.* Prentice-Hall, Englewood Cliffs, New Jersey, 1985.